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The mixed mode crack problem in an FGM layer bonded to a homogeneous half-plane ☆

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Abstract

In this paper, the plane elasticity problem of an arbitrarily oriented crack in a FGM layer bonded to a homogeneous half-plane is considered. The problem is modeled by assuming that the elastic properties of the FGM layer are exponential functions of the thickness coordinate and are continuous at the interface of the FGM layer and the half-plane.

The Fourier transform technique is used to reduce the problem to the solution of a system of Cauchy-type singular integral equations, which are solved numerically. The stress intensity factors are computed for various crack orientations, crack locations and material parameters. The results show that crack length, crack orientation and the non-homogeneity parameter of the strip material have significant effect on the fracture of the FGM layer.

1. Introduction

In the recent past, there has been a strong increase in interest in functionally graded materials (FGMs). The idea of Functionally Graded Materials initially came from the need of more efficient combustion process in many high temperature aerospace applications such as turbines, compressors and combustion chambers. FGMs are essentially two-phase particulate composites whose composition, microstructure and properties vary gradually. Most FGMs are made from ceramics and metals. Ceramics provide thermal and corrosion resistance while metals provide the necessary mechanical toughness and heat conductivity. The volume fractions of the constituents in an FGM usually vary continuously from 100% ceramic at

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the surface to 0% at the interface continuously. The conditions for both high temperature and high toughness can then be met simultaneously. FGMs can be tailored to meet the rigorous requirements encountered in practice through the design of their constituents. They have great potential for applications in safety-critical structures such as nuclear fusion reactors, aircraft fuselages, microelectronic devices and biomaterials, but the greatest area of focus and initial emphasis appear to be in thermal barrier protection for turbines engines, etc. in high temperature environments.

From the mechanical point of view, FGMs are unique because their mechanical properties vary spatially. This distinctive feature that makes FGMs such promising candidates for advanced technological applications, happens to create enormous difficulties for those who want to analytically study the fracture behavior of FGMs. It is well known that if the mechanical properties of the material are not constant, the governing elasticity equations become partial differential equations with variable coefficients, which complicate the analysis significantly. Except for few idealized cases, most crack problems for FGMs are not amenable to analytical treatments because of their intrinsic complexities.

Some fracture problems of FGMs have been investigated in the past decade. Delale and Erdogan (1983) studied mode I crack problem in an infinite non-homogeneous plane, and found that the effect of the Poisson's ratio on the stress intensity factors is negligible. Konda and Erdogan (1994) solved the more general case of mixed mode crack problem in FGMs. Erdogan and Wu (1997) investigated the mode I crack problem in a FGM strip, especially for edge cracks. The mode I crack parallel to the boundary of an infinite strip was solved by El-Borgi et al. (2000), with the Young's modulus varying exponentially in an arbitrary direction. Long and Delale (2004) studied the general problem for an unconstrained FGM layer containing an arbitrarily oriented crack.

Cracks in multilayer materials with at least one layer being nonhomogeneous are studied less extensively, and most of these works are limited to cracks in the homogeneous planes or interfacial region (Chen and Erdogan, 1996; Choi, 1996; Choi, 2001; Delale, 1985; Delale and Erdogan, 1988a,b; Erdogan, 1985; Erdogan et al., 1991; Erdogan et al., 1991; Schovanec and Walton, 1988; Shbeeb and Binienda, 1999; Ueda, 2001). Choi (2001) studied an arbitrarily oriented crack located in a homogeneous semi-infinite substrate that is bonded to a surface layer through a nonhomogeneous interfacial layer, which is the closest to the problem studied in the current paper.

In this paper, the more general problem of an arbitrarily oriented crack in a FGM layer bonded to a homogeneous half-plane is studied. The problem is a model for a functionally graded TCB bonded to a substrate. First, some auxiliary functions are introduced and then the problem is formulated in terms of a system of Cauchy-type singular integral equations. These equations are solved numerically to obtain the stress intensity factors at the crack tips.

To make the problem mathematically tractable, the Young's modulus of the material is assumed to vary exponentially in the thickness coordinate. The Poisson's ratio is assumed to be constant.

2. The formulation

The crack problem under consideration is an FGM strip (material 1 in Fig. 1) of thickness h containing an embedded finite crack on the $y' = 0$ plane; the strip is bonded to a homogeneous half-plane (material 2 in Fig. 1). In the FGM strip, the material properties vary exponentially in the thickness direction. The Poisson's ratio ν is constant and the shear modulus is defined by:

$$\mu(x) = \mu_1 e^{\delta x} \quad \text{or} \quad \mu(x', y') = \mu_1 e^{\beta x' + \gamma y'} \tag{1}$$

where,

$$\beta = \delta \cos\theta, \quad \gamma = -\delta \sin\theta. \tag{2}$$

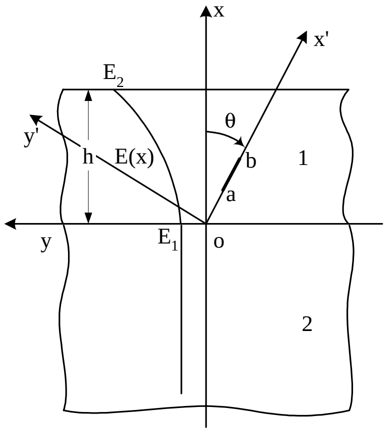

Fig. 1. Crack geometry of the problem.

The Lame constant $\lambda(x,y)$ can be written as

$$\lambda(x,y) = \frac{3-\kappa}{\kappa-1}\mu_1 e^{\beta x'+\gamma y'} \tag{3}$$

with

$$\kappa = \begin{cases} 3-4v & \text{for plane strain} \\ \dfrac{3-v}{1+v} & \text{for plane stress} \end{cases} \tag{4}$$

Here, δ is a constant that describes the nonhomogeneity of the FGM. To simplify the discussion, we assume $\delta \geqslant 0$. μ_1 is the share modulus at $x = 0$. θ is the angle between the crack line and x.

The material properties of the half-plane are the same as those of the FGM layer at the interface

$$E(x) = E_1; \quad \mu(x) = \mu_1 \tag{5}$$

The problem will be solved under the following boundary and continuity conditions

$$\left.\begin{aligned} \sigma_{y'}(x',+0) &= \sigma_{y'}(x',-0) \\ \tau_{x'y'}(x',+0) &= \tau_{x'y'}(x',-0) \end{aligned}\right\} \quad 0 < x < h \tag{6}$$

$$\left.\begin{aligned} \sigma_x(+0,y) &= \sigma_x(-0,y) \\ \tau_{xy}(+0,y) &= \tau_{xy}(-0,y) \end{aligned}\right\} \quad -\infty < y < \infty \tag{7}$$

$$\sigma_x(h,y) = \tau_{xy}(h,y) = 0 \quad -\infty < y < \infty \tag{8}$$

$$\left.\begin{aligned} v(x',+0) &= v(x',-0) \\ u(x',+0) &= u(x',-0) \end{aligned}\right\} \quad x' < a \quad \text{or} \quad x' > b \quad \text{and} \quad 0 < x < h \tag{9}$$

$$\left.\begin{aligned} v(+0,y) &= v(-0,y) \\ u(+0,y) &= u(-0,y) \end{aligned}\right\} \quad -\infty < y < \infty \tag{10}$$

$$\left.\begin{aligned}\sigma_{y'}(x',0)=p_1(x')\\ \tau_{x'y'}(x',0)=p_2(x')\end{aligned}\right\}\quad a<x'<b \tag{11}$$

Here, $p_1(x')$ and $p_2(x')$ are given crack surface tractions, which can be determined by solving the elasticity problem for the uncracked strip under the given external loads.

2.1. Basic solution for the FGM layer

The solution of the FGM layer is expressed as the sum of two states of strain as $u_1'(x',y'),v_1'(x',y')$ and $u_1(x,y)$ and $v_1(x,y)$ where the coordinates (x',y') and (x,y) are defined in Fig. 1.

The governing equations for the FGM plane may be expressed as

$$\begin{aligned}&(\kappa+1)\frac{\partial^2u_1'}{\partial x'^2}+(\kappa-1)\frac{\partial^2u_1'}{\partial y'^2}+2\frac{\partial^2v_1'}{\partial x'\partial y'}+\beta(\kappa+1)\frac{\partial u_1'}{\partial x'}+\gamma(\kappa-1)\left(\frac{\partial u_1'}{\partial y'}+\frac{\partial v_1'}{\partial x'}\right)+\beta(3-\kappa)\frac{\partial v_1'}{\partial y'}=0\\ &(\kappa-1)\frac{\partial^2v_1'}{\partial x'^2}+(\kappa+1)\frac{\partial^2v_1'}{\partial y'^2}+2\frac{\partial^2u_1'}{\partial x'\partial y'}+\gamma(3-\kappa)\frac{\partial u_1'}{\partial x'}+\beta(\kappa-1)\left(\frac{\partial u_1'}{\partial y'}+\frac{\partial v_1'}{\partial x'}\right)+\gamma(\kappa+1)\frac{\partial v_1'}{\partial y'}=0\end{aligned} \tag{12}$$

Assuming, $u_1'(x',y')$ and $v_1'(x',y')$ as

$$\begin{aligned}u_1'(x',y')&=\frac{1}{2\pi}\int_{-\infty}^{\infty}U(y',\alpha)\mathrm{e}^{-\mathrm{i}\alpha x'}\,\mathrm{d}\alpha\\ v_1'(x',y')&=\frac{1}{2\pi}\int_{-\infty}^{\infty}V(y',\alpha)\mathrm{e}^{-\mathrm{i}\alpha x'}\,\mathrm{d}\alpha\end{aligned} \tag{13}$$

Substituting expressions (13) into Eqs. (12) and after some manipulations, we get

$$U(y',\alpha)=\sum_{j=1}^{4}m_jF_j(\alpha)\mathrm{e}^{n_jy'}\quad V(y',\alpha)=\sum_{j=1}^{4}F_j(\alpha)\mathrm{e}^{n_jy'} \tag{14}$$

where $F_j(\alpha)$ are unknown functions and m_j $(j=1,\ldots,4)$ are given by

$$m_j=\frac{[2\alpha\mathrm{i}+\beta(\kappa-3)]n_j+\mathrm{i}\alpha\gamma(\kappa-1)}{(\kappa-1)n_j^2+(\kappa-1)\gamma n_j-(\kappa+1)(\alpha+\mathrm{i}\beta)\alpha}\quad j=1,\ldots,4 \tag{15}$$

while n_j $(j=1,\ldots,4)$ are the roots of

$$[n^2+\gamma n-\alpha(a+\mathrm{i}\beta)]^2+\frac{3-\kappa}{\kappa+1}(\alpha\gamma-\mathrm{i}\beta n)^2=0 \tag{16}$$

The values of n_j are

$$\begin{aligned}n_1&=-\frac{\Delta_1}{2}-\frac{\sqrt{\Delta_1^2+4(\alpha^2+\mathrm{i}\alpha\Delta_2)}}{2}\quad n_2=-\frac{\Delta_3}{2}-\frac{\sqrt{\Delta_3^2+4(\alpha^2+\mathrm{i}\alpha\Delta_4)}}{2}\\ n_3&=-\frac{\Delta_1}{2}+\frac{\sqrt{\Delta_1^2+4(\alpha^2+\mathrm{i}\alpha\Delta_2)}}{2}\quad n_4=-\frac{\Delta_3}{2}+\frac{\sqrt{\Delta_3^2+4(\alpha^2+\mathrm{i}\alpha\Delta_4)}}{2}\end{aligned} \tag{17}$$

where

$$\begin{aligned}\Delta_1&=\gamma+\beta\sqrt{\frac{3-\kappa}{\kappa+1}}\quad \Delta_3=\gamma-\beta\sqrt{\frac{3-\kappa}{\kappa+1}}\\ \Delta_2&=\beta-\gamma\sqrt{\frac{3-\kappa}{\kappa+1}}\quad \Delta_4=\beta+\gamma\sqrt{\frac{3-\kappa}{\kappa+1}}\end{aligned} \tag{18}$$

Since u_1' and v_1' must vanish for $x^2 + y^2 \to \infty$, we have

$$\begin{aligned} &F_3(\alpha) = F_4(\alpha) = 0, \quad y > 0 \\ &F_1(\alpha) = F_2(\alpha) = 0, \quad y < 0 \end{aligned} \tag{19}$$

Using generalized Hooke's Law, we obtain

$$\begin{aligned} \sigma'_{x'}(x', y') &= \frac{\mu}{2\pi(\kappa - 1)} \int_{-\infty}^{\infty} \sum_{j=l}^{l+1} [-\mathrm{i}\alpha m_j(1+\kappa) + n_j(3-\kappa)] F_j(\alpha) \mathrm{e}^{n_j y' - \mathrm{i}\alpha x'} \, \mathrm{d}\alpha \\ \sigma'_{y'}(x', y') &= \frac{\mu}{2\pi(\kappa - 1)} \int_{-\infty}^{\infty} \sum_{j=l}^{l+1} [-\mathrm{i}\alpha m_j(3-\kappa) + n_j(1+\kappa)] F_j(\alpha) \mathrm{e}^{n_j y' - \mathrm{i}\alpha x'} \, \mathrm{d}\alpha \\ \tau'_{x'y'}(x', y') &= \frac{\mu}{2\pi} \int_{-\infty}^{\infty} \sum_{j=l}^{l+1} [n_j m_j - \mathrm{i}\alpha] F_j(\alpha) \mathrm{e}^{n_j y' - \mathrm{i}\alpha x'} \, \mathrm{d}\alpha \end{aligned} \tag{20}$$

where $l = 1$ for $y > 0$ and $l = 3$ for $y < 0$.

From the continuity conditions (6) and Eqs. (20), we find

$$F_3(\alpha) = R_1(\alpha)F_1(\alpha) + R_2(\alpha)F_2(\alpha), \quad F_4(\alpha) = R_3(\alpha)F_1(\alpha) + R_4(\alpha)F_2(\alpha) \tag{21}$$

where $R_j(\alpha)$ are known functions given in Appendix A.

We then introduce the following auxiliary functions:

$$\begin{aligned} g_1(x') &= \frac{\partial}{\partial x'} [u_1'(x', +0) - u_1'(x', -0)], \quad a < | x' | < b \\ g_2(x') &= \frac{\partial}{\partial x'} [v_1'(x', +0) - v_1'(x', -0)], \quad a < | x' | < b \end{aligned} \tag{22}$$

We can express F_j ($j = 1, \ldots, 4$) in terms of these auxiliary functions as

$$\begin{aligned} F_1 &= \int_{-\infty}^{\infty} \frac{[\alpha(3-\kappa)f_{22} + \mathrm{i}(1+\kappa)f_{32}]g_1 + (\mathrm{i}f_{12} + \alpha f_{42})(1+\kappa)g_2}{(1+\kappa)\alpha\omega_0} \mathrm{e}^{\mathrm{i}\alpha t} \, \mathrm{d}t \\ F_2 &= -\int_{-\infty}^{\infty} \frac{[\alpha(3-\kappa)f_{21} + i(1+\kappa)f_{31}]g_1 + (\mathrm{i}f_{11} + \alpha f_{41})(1+\kappa)g_2}{(1+\kappa)\alpha\omega_0} \mathrm{e}^{\mathrm{i}\alpha t} \, \mathrm{d}t \\ F_3 &= \int_{-\infty}^{\infty} \frac{[\alpha(3-\kappa)f_{24} + \mathrm{i}(1+\kappa)f_{34}]g_1 + (\mathrm{i}f_{14} + \alpha f_{44})(1+\kappa)g_2}{(1+\kappa)\alpha\omega_0} \mathrm{e}^{\mathrm{i}\alpha t} \, \mathrm{d}t \\ F_4 &= -\int_{-\infty}^{\infty} \frac{[\alpha(3-\kappa)f_{23} + \mathrm{i}(1+\kappa)f_{33}]g_1 + (if_{13} + \alpha f_{43})(1+\kappa)g_2}{(1+\kappa)\alpha\omega_0} \mathrm{e}^{\mathrm{i}\alpha t} \, \mathrm{d}t \end{aligned} \tag{23}$$

where $f_{i,j}$ ($i = 1, \ldots, 4$; $j = 1, \ldots, 4$) are known functions given in Appendix A, and

$$\begin{aligned} \omega_0 = {} & (m_1 - m_2)(m_3 - m_4)(n_1 n_2 + n_3 n_4) + (m_1 - m_4)(m_2 - m_3)(n_2 n_3 + n_1 n_4) \\ & - (m_1 - m_3)(m_2 - m_4)(n_1 n_3 + n_2 n_4) \end{aligned} \tag{24}$$

Substituting F_j ($j = 1, \ldots, 4$) back into Eq. (20), we can express $\sigma'_{x'}$, $\sigma'_{y'}$ and $\tau'_{x'y'}$ in terms of g_1 and g_2, for $y' > 0$, as

$$
\begin{aligned}
\sigma_{x'}^{(1)}(x',y') &= \frac{\mu(x',y')}{2\pi(1+\kappa)}\int_a^b \sum_{j=1}^{2} h_{1j}^{(1)}(x',y',t)g_j(t)\,\mathrm{d}t \\
\sigma_{y'}^{(1)}(x',y') &= \frac{\mu(x',y')}{2\pi(1+\kappa)}\int_a^b \sum_{j=1}^{2} h_{2j}^{(1)}(x',y',t)g_j(t)\,\mathrm{d}t \\
\tau_{x'y'}^{(1)}(x',y') &= \frac{\mu(x',y')}{2\pi(1+\kappa)}\int_a^b \sum_{j=1}^{2} h_{3j}^{(1)}(x',y',t)g_j(t)\,\mathrm{d}t
\end{aligned} \tag{25}
$$

where

$$
h_{kj}^{(1)}(x',y',t) = \int_{-\infty}^{\infty} K_{kj}^{(1)}(y',\alpha)\mathrm{e}^{\mathrm{i}\alpha(t-x')}\,\mathrm{d}\alpha, \quad k=1,2;\;\; j=1,2 \tag{26}
$$

and $K_{kj}^{(1)}(y',\alpha)$ $(k=1,2; j=1,2)$ are known functions given in Appendix A.

Similarly, when $y'<0$, $\sigma'_{x'}$, $\sigma'_{y'}$ and $\tau'_{x'y'}$ can be written as

$$
\begin{aligned}
\sigma_{x'}^{(2)}(x',y') &= \frac{\mu(x',y')}{2\pi(1+\kappa)}\int_a^b \sum_{j=1}^{2} h_{1j}^{(2)}(x',y',t)g_j(t)\,\mathrm{d}t \\
\sigma_{y'}^{(2)}(x',y') &= \frac{\mu(x',y')}{2\pi(1+\kappa)}\int_a^b \sum_{j=1}^{2} h_{2j}^{(2)}(x',y',t)g_j(t)\,\mathrm{d}t \\
\tau_{x'y'}^{(2)}(x',y') &= \frac{\mu(x',y')}{2\pi(1+\kappa)}\int_a^b \sum_{j=1}^{2} h_{3j}^{(2)}(x',y',t)g_j(t)\,\mathrm{d}t
\end{aligned} \tag{27}
$$

where

$$
h_{kj}^{(2)}(x',y',t) = \int_{-\infty}^{\infty} K_{kj}^{(2)}(y',\alpha)\mathrm{e}^{\mathrm{i}\alpha(t-x')}\,\mathrm{d}\alpha, \quad k=1,2; \quad j=1,2,
$$

and $K_{kj}^{(2)}(y',\alpha)(k=1,2;\; j=1,2)$ are known functions given in the Appendix A.

In the (x,y) coordinate system, the Navier's equations for the FGM layer may be expressed as

$$
\begin{aligned}
&(\kappa+1)\frac{\partial^2 u_1}{\partial x^2} + (\kappa-1)\frac{\partial^2 u_1}{\partial y^2} + 2\frac{\partial^2 v_1}{\partial x\partial y} + \delta(\kappa+1)\frac{\partial u_1}{\partial x} + \delta(3-\kappa)\frac{\partial v_1}{\partial y} = 0 \\
&(\kappa-1)\frac{\partial^2 v_1}{\partial x^2} + (\kappa+1)\frac{\partial^2 v_1}{\partial y^2} + 2\frac{\partial^2 u_1}{\partial x\partial y} + \delta(\kappa-1)\left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y}\right) = 0
\end{aligned} \tag{28}
$$

Assuming

$$
\begin{aligned}
u_1(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} qA(\alpha)\mathrm{e}^{px}\mathrm{e}^{-\mathrm{i}\alpha y}\,\mathrm{d}\alpha \\
v_1(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\alpha)\mathrm{e}^{px}\mathrm{e}^{-\mathrm{i}\alpha y}\,\mathrm{d}\alpha
\end{aligned} \tag{29}
$$

and substituting Eqs. (29) into (28), we obtain the characteristic equation as

$$
\begin{aligned}
&[(\kappa+1)p^2 - (\kappa-1)\alpha^2 + \delta(\kappa+1)p]q - \mathrm{i}\alpha[2p + \delta(3-\kappa)] = 0 \quad (a) \\
&(\kappa-1)p^2 + \delta(\kappa-1)p - (\kappa+1)\alpha^2 - \mathrm{i}\alpha q[2p + \delta(\kappa-1)] = 0 \quad (b)
\end{aligned} \tag{30}
$$

solving Eq. (30), we have:

$$q_j = \frac{(\kappa-1)(p_j+\delta)p_j - (\kappa+1)\alpha^2}{\alpha\mathrm{i}[2p_j+\delta(\kappa-1)]}, \quad j = 1,\ldots,4 \tag{31}$$

Defining $\omega = \delta\sqrt{(3-\kappa)/(1+\kappa)}$, and using Eq. (31), Eq. (30) yield the roots p as

$$\begin{aligned}
p_1 &= -\frac{\delta}{2} - \frac{1}{2}\sqrt{\delta^2 + 4\alpha^2 + 4\mathrm{i}\omega\alpha}\\
p_2 &= -\frac{\delta}{2} - \frac{1}{2}\sqrt{\delta^2 + 4\alpha^2 - 4\mathrm{i}\omega\alpha}\\
p_3 &= -\frac{\delta}{2} + \frac{1}{2}\sqrt{\delta^2 + 4\alpha^2 + 4\mathrm{i}\omega\alpha}\\
p_4 &= -\frac{\delta}{2} + \frac{1}{2}\sqrt{\delta^2 + 4\alpha^2 - 4\mathrm{i}\omega\alpha}
\end{aligned} \tag{32}$$

Thus, Eqs. (29) become

$$\begin{aligned}
u_1(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_j^4 q_j A_j(\alpha)\mathrm{e}^{p_j x}\mathrm{e}^{-\mathrm{i}\alpha y}\,\mathrm{d}\alpha\\
v_1(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_j^4 A_j(\alpha)\mathrm{e}^{p_j x}\mathrm{e}^{-\mathrm{i}\alpha y}\,\mathrm{d}\alpha
\end{aligned} \tag{33}$$

Then, the corresponding stresses are obtained as

$$\begin{aligned}
\sigma_x^{(3)} &= \frac{\mu(x,y)}{2\pi(\kappa-1)}\int_{-\infty}^{\infty}\sum_j^4[(1+k)p_j q_j + (\kappa-3)\mathrm{i}\alpha]A_j(\alpha)\mathrm{e}^{p_j x - \mathrm{i}\alpha y}\,\mathrm{d}\alpha\\
\sigma_y^{(3)} &= \frac{\mu(x,y)}{2\pi(\kappa-1)}\int_{-\infty}^{\infty}\sum_j^4[-(1+k)\mathrm{i}\alpha + (3-\kappa)p_j q_j]A_j(\alpha)\mathrm{e}^{p_j x - \mathrm{i}\alpha y}\,\mathrm{d}\alpha\\
\tau_{xy}^{(3)} &= \frac{\mu(x,y)}{2\pi}\int_{-\infty}^{\infty}\sum_j^4[p_j - \mathrm{i}\alpha q_j]A_j(\alpha)\mathrm{e}^{p_j x - \mathrm{i}\alpha y}\,\mathrm{d}\alpha
\end{aligned} \tag{34}$$

The stress state at a given point in the FGM plane can be expressed as the sum of the stresses given by Eqs. (34) and Eqs. (25) or (27), depending on the sign of y' at that point.

From the free boundary conditions (8) of the problem at $x = h$, we have

$$\sum_j^4[(1+k)p_j q_j + (\kappa-3)\mathrm{i}\alpha]A_j(\alpha)\mathrm{e}^{p_j h} = \frac{1-\kappa}{2\pi(1+\kappa)}\sum_{j=1}^{2}\int_a^b Q_{1j}(\alpha,t)g_j(t)\,\mathrm{d}t \tag{35}$$

where

$$Q_{1j}(\alpha,t) = \sum_{i=1}^{2}\{\xi_{1j}^{(i)}(\alpha,t)\cos^2\theta + \xi_{2j}^{(i)}(\alpha,t)\sin^2\theta - 2\xi_{3j}^{(i)}(\alpha,t)\cos\theta\sin\theta\} \tag{36}$$

$$\begin{aligned}
\xi_{kj}^{(1)}(\alpha,t) &= \int_{-\infty}^{\infty}\int_{h\tan\theta}^{\infty} K_{kj}^{(1)}(y\cos\theta - h\sin\theta,\rho)\mathrm{e}^{\mathrm{i}\rho(t-y\sin\theta-h\cos\theta)+\mathrm{i}\alpha y}\,\mathrm{d}y\,\mathrm{d}\rho\\
\xi_{kj}^{(2)}(\alpha,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{h\tan\theta} K_{kj}^{(2)}(y\cos\theta - h\sin\theta,\rho)\mathrm{e}^{\mathrm{i}\rho(t-y\sin\theta-h\cos\theta)+\mathrm{i}\alpha y}\,\mathrm{d}y\,\mathrm{d}\rho
\end{aligned} \tag{37}$$

and

$$\sum_{j}^{4}[p_j - \mathrm{i}\alpha q_j]A_j(\alpha)\mathrm{e}^{p_j h} = \frac{-1}{2\pi(1+\kappa)}\sum_{j=1}^{2}\int_a^b Q_{2j}(\alpha,t)g_j(t)\,\mathrm{d}t \tag{38}$$

where

$$Q_{2j}(\alpha,t) = \sum_{i=1}^{2}\left\{(\cos^2\theta - \sin^2\theta)\xi_{3j}^{(i)}(\alpha,t) - [\xi_{2j}^{(i)}(\alpha,t) - \xi_{1j}^{(i)}(\alpha,t)]\sin\theta\cos\theta\right\} \tag{39}$$

2.2. The solution for the half-plane

For the half-plane, Navier's equations in terms of displacements u_2 and v_2 may be expressed as

$$\begin{aligned}&(\kappa+1)\frac{\partial^2 u_2}{\partial x^2} + (\kappa-1)\frac{\partial^2 u_2}{\partial y^2} + 2\frac{\partial^2 v_2}{\partial x\partial y} = 0\\&(\kappa-1)\frac{\partial^2 v_2}{\partial x^2} + (\kappa+1)\frac{\partial^2 v_2}{\partial y^2} + 2\frac{\partial^2 u_2}{\partial x\partial y} = 0\end{aligned} \tag{40}$$

For $y < 0$, assuming $u_2(x,y)$ and $v_2(x,y)$ in the following form:

$$\begin{aligned}u_2(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_1(x,\alpha)\mathrm{e}^{-\mathrm{i}\alpha y}\,\mathrm{d}\alpha\\v_2(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_2(x,\alpha)\mathrm{e}^{-\mathrm{i}\alpha y}\,\mathrm{d}\alpha\end{aligned} \tag{41}$$

and substituting into (40), we get

$$\begin{aligned}u_2(x,y) &= \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\alpha}{|\alpha|}\left[G_1(\alpha) + \left(x - \frac{\kappa}{|\alpha|}\right)G_2(\alpha)\right]\mathrm{e}^{|\alpha|x - \mathrm{i}\alpha y}\,\mathrm{d}\alpha\\v_2(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\alpha) + xG_2(\alpha)\mathrm{e}^{|\alpha|x - \mathrm{i}\alpha y}\,\mathrm{d}\alpha\end{aligned} \tag{42}$$

From generalized Hooke's law, the stresses for the half-plane are obtained as

$$\begin{aligned}\sigma_{2x} &= \frac{\mu_1\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\left[2\alpha(G_1 + xG_2) - \frac{\alpha}{|\alpha|}(1+\kappa)G_2\right]\mathrm{e}^{|\alpha|x - \mathrm{i}\alpha y}\,\mathrm{d}\alpha\\\sigma_{2y} &= -\frac{\mu_1\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\left[2\alpha(G_1 + xG_2) + \frac{\alpha}{|\alpha|}(3-\kappa)G_2\right]\mathrm{e}^{|\alpha|x - \mathrm{i}\alpha y}\,\mathrm{d}\alpha\\\tau_{2xy} &= \frac{\mu_1}{2\pi}\int_{-\infty}^{\infty}[2\,|\alpha|\,(G_1 + xG_2) + (1-\kappa)G_2]\mathrm{e}^{|\alpha|x - \mathrm{i}\alpha y}\,\mathrm{d}\alpha\end{aligned} \tag{43}$$

3. The integral equations

Using the continuity conditions (7) and (10), we get following relations:

$$\sum_j^4 [(1+k)p_j q_j + (\kappa-3)\mathrm{i}\alpha]A_j(\alpha) - \mathrm{i}\left[2\alpha G_1 - \frac{\alpha}{|\alpha|}(1+\kappa)G_2\right](\kappa-1) = \frac{1-\kappa}{2\pi(1+\kappa)}\sum_{j=1}^2\int_a^b Q_{3j}(\alpha,t)g_j(t)\,\mathrm{d}t \tag{44}$$

$$\sum_j^4 [p_j - \mathrm{i}\alpha q_j]A_j(\alpha) - [2\,|\,\alpha\,|\,G_1 + (1-\kappa)G_2] = \frac{-1}{2\pi(1+\kappa)}\sum_{j=1}^2\int_a^b Q_{4j}g_j(t)\,\mathrm{d}t \tag{45}$$

$$\sum_j^4 q_j A_j(\alpha) - \frac{\alpha \mathrm{i}}{|\alpha|}\left[G_1(\alpha) - \frac{\kappa}{|\alpha|}G_2(\alpha)\right] = \sum_{j=1}^2\int_a^b Q_{5j}(\alpha,t)g_j(t)\,\mathrm{d}t \tag{46}$$

$$\sum_j^4 A_j(\alpha)\mathrm{e}^{-\mathrm{i}\alpha y} - G_1(\alpha) = \sum_{j=1}^2\int_a^b Q_{6j}(\alpha,t)g_j(t)\,\mathrm{d}t \tag{47}$$

where $Q_{kj}(\alpha,t)$ $(k=3,\ldots,6;\ j=1,2)$ are known functions shown in Appendix A.

With Eqs. (35), (38), (44), (45), (46) and (47) we have six equations for the six unknowns $A_j (j=1,\ldots,4)$ and $G_j (j=1,2)$:

$$\begin{aligned}
&\sum_j^4 [(1+k)p_j q_j + (\kappa-3)\mathrm{i}\alpha]A_j(\alpha)\mathrm{e}^{p_j h} = \frac{1-\kappa}{2\pi(1+\kappa)}\sum_{j=1}^2\int_a^b Q_{1j}(\alpha,t)g_j(t)\,\mathrm{d}t\\
&\sum_j^4 [p_j - \mathrm{i}\alpha q_j]A_j(\alpha)\mathrm{e}^{p_j h} = \frac{-1}{2\pi(1+\kappa)}\sum_{j=1}^2\int_a^b Q_{2j}(\alpha,t)g_j(t)\,\mathrm{d}t\\
&\sum_j^4 [(1+k)p_j q_j + (\kappa-3)\mathrm{i}\alpha]A_j(\alpha) - \mathrm{i}\left[2\alpha G_1 - \frac{\alpha}{|\alpha|}(1+\kappa)G_2\right](\kappa-1)\\
&\qquad = \frac{1-\kappa}{2\pi(1+\kappa)}\sum_{j=1}^2\int_a^b Q_{3j}(\alpha,t)g_j(t)\,\mathrm{d}t\\
&\sum_j^4 [p_j - \mathrm{i}\alpha q_j]A_j(\alpha) - [2\,|\,\alpha\,|\,G_1 + (1-\kappa)G_2] = \frac{-1}{2\pi(1+\kappa)}\sum_{j=1}^2\int_a^b Q_{4j}g_j(t)\,\mathrm{d}t\\
&\sum_j^4 q_j A_j(\alpha) - \frac{\alpha \mathrm{i}}{|\alpha|}\left[G_1(\alpha) - \frac{\kappa}{|\alpha|}G_2(\alpha)\right] = \sum_{j=1}^2\int_a^b Q_{5j}(\alpha,t)g_j(t)\,\mathrm{d}t\\
&\sum_j^4 A_j(\alpha)\mathrm{e}^{-\mathrm{i}\alpha y} - G_1(\alpha) = \sum_{j=1}^2\int_a^b Q_{6j}(\alpha,t)g_j(t)\,\mathrm{d}t
\end{aligned} \tag{48}$$

Solving Eqs. (48) we can obtain $A_j(\alpha)$ $(j=1,\ldots,4)$ and $G_1(\alpha)$, $G_2(\alpha)$.

From the boundary conditions

$$\sigma_{y_1}(x_1,+0) = p_1(x_1), \quad \tau_{x_1 y_1}(x_1,+0) = p_2(x_1) \qquad (a<x<b) \tag{49}$$

where $p_1(x_1)$ and $p_2(x_1)$ are the crack surface tractions, we obtain

$$\begin{aligned} &\sigma_{y_1}^{(1)}(x_1,0)+\sin^2\theta\sigma_x^{(3)}(x_1\cos\theta,x_1\sin\theta)+\cos^2\theta\sigma_y^{(3)}(x_1\cos\theta,x_1\sin\theta)\\ &\quad-2\sin\theta\cos\theta\tau_{xy}^{(3)}(x_1\cos\theta,x_1\sin\theta)=p_1(x_1)\\ &au_{x_1y_1}^{(1)}(x_1,0)+\tau_{xy}^{(3)}(x_1\cos\theta,x_1\sin\theta)(\cos^2\theta-\sin^2\theta)\\ &\quad+\sin\theta\cos\theta[\sigma_y^{(3)}(x_1\cos\theta,x_1\sin\theta)-\sigma_x^{(3)}(x_1\cos\theta,x_1\sin\theta)]=p_2(x_1) \end{aligned} \tag{50}$$

To obtain the asymptotic behavior of the stresses, first we rewrite the first equation of (50) as

$$\begin{aligned} &\frac{\mu(x_1,0)}{2\pi(1+\kappa)}\int_a^b\sum_{i=1}^{2}h_{2j}^{(1)}(x_1,t)g_i(t)\,\mathrm{d}t+\frac{\mu(x_1\cos\theta,x_1\sin\theta)}{2\pi(\kappa-1)}\int_{-\infty}^{\infty}\sum_j^4\{[(1+k)p_jq_j\\ &\quad+(\kappa-3)\mathrm{i}\alpha]\sin^2\theta+[-(1+k)\mathrm{i}\alpha+(3-\kappa)p_jq_j]\cos^2\theta\\ &\quad-2\sin\theta\cos\theta(\kappa-1)[p_j-\mathrm{i}\alpha q_j]\}A_j(\alpha)\mathrm{e}^{p_jx_1\cos\theta-\mathrm{i}\alpha x_1\sin\theta}\,\mathrm{d}\alpha=p_1(x_1) \end{aligned} \tag{51}$$

Since the asymptotic value of K_{21} and K_{22} for $\alpha\to\infty$ are

$$\begin{aligned} &K_{21}^{\infty}=0\\ &K_{22}^{\infty}=-2\mathrm{i}\frac{|\alpha|}{\alpha}\mathrm{e}^{-|\alpha|y} \end{aligned} \tag{52}$$

Eq. (51) yields,

$$\frac{1}{\pi}\int_a^b\left\{\frac{g_2(t)}{t-x_1}+\sum_{j=1}^{2}[k_{1j}^{(1)}(x_1,t)+k_{2j}^{(1)}(x_1,t)]g_j(t)\right\}\mathrm{d}t=\frac{(1+\kappa)}{2\mu(x_1,0)}p_1(x_1) \tag{53}$$

with the kernels

$$\begin{aligned} &k_{11}^{(1)}(x_1,t)=\frac{1}{4}h_{21}^{(1)}(x_1,0,t)\\ &k_{12}^{(1)}(x_1,t)=\frac{1}{4}\int_{-\infty}^{\infty}[K_{22}^{(1)}(0,\alpha)-\lim_{\alpha\to\infty}K_{22}^{(1)}(0,\alpha)]\mathrm{e}^{\mathrm{i}\alpha(t-x_1)}\,\mathrm{d}\alpha\\ &k_{2j}^{(2)}(x_1,t)=\frac{(1+\kappa)}{4(\kappa-1)}\int_{-\infty}^{\infty}\sum_i^4\{[(1+k)p_iq_i+(\kappa-3)\mathrm{i}\alpha]\sin^2\theta\\ &\quad+[-(1+k)\mathrm{i}\alpha+(3-\kappa)p_iq_i]\cos^2\theta\\ &\quad-2\sin\theta\cos\theta(\kappa-1)[p_i-\mathrm{i}\alpha q_i]\}C_{ij}(\alpha,t)\mathrm{e}^{p_ix_1\cos\theta-\mathrm{i}\alpha x_1\sin\theta}\,\mathrm{d}\alpha \end{aligned} \tag{54}$$

Similarly, the second equation of Eq. (50) becomes

$$\frac{1}{\pi}\int_a^b\left\{\frac{g_1(t)}{t-x_1}+\sum_{j=1}^{2}[k_{1j}^{(2)}(x_1,t)+k_{2j}^{(2)}(x_1,t)]g_j(t)\right\}\mathrm{d}t=\frac{(1+\kappa)}{2\mu(x_1,0)}p_2(x_1) \tag{55}$$

where

$$\begin{aligned} &k_{12}^{(2)}(x_1,t)=\frac{1}{4}h_{32}^{(1)}(x_1,0,t)\\ &k_{11}^{(2)}(x_1,t)=\frac{1}{4}\int_{-\infty}^{\infty}[K_{31}^{(1)}(0,\alpha)-\lim_{\alpha\to\infty}K_{31}^{(1)}(0,\alpha)]\mathrm{e}^{\mathrm{i}\alpha(t-x_1)}\,\mathrm{d}\alpha\\ &k_{2i}^{(2)}(x_1,t)=\frac{(1+\kappa)}{4(\kappa-1)}\int_{-\infty}^{\infty}\sum_j^4\{(\cos^2\theta-\sin^2\theta)(\kappa-1)[p_j-\mathrm{i}\alpha q_j]\\ &\quad+2\sin\theta\cos\theta(1-\kappa)[\mathrm{i}\alpha+p_jq_j]\}C_{ji}(\alpha,t)\mathrm{e}^{p_jx_1\cos\theta-\mathrm{i}\alpha x_1\sin\theta}\,\mathrm{d}\alpha \end{aligned} \tag{56}$$

From Eqs. (22), we obtain the single valuedness conditions to complete the formulation of the problem

$$\int_a^b g_j(t)\,\mathrm{d}t = 0, \quad j = 1,2 \tag{57}$$

4. The numerical solution

To obtain stress intensity factors at the crack tips, the Cauchy-type singular integral equations are solved numerically.

First, we define

$$\begin{aligned}
&t = \frac{b-a}{2}r + \frac{b+a}{2} \\
&x_1 = \frac{b-a}{2}s + \frac{b+a}{2} \\
&g_1(t) = \phi_1(r) \quad g_2(t) = \phi_2(r) \\
&p_1(x_1) = f_1(s) \quad p_2(x_1) = f_2(s) \\
&\mu(x_1,0) = m(s,0) \\
&q_{ij}^{(n)}(s,r) = \frac{b-a}{2}k_{ij}^{(n)}(x_1,t) \quad (i=1,2,\ j=1,2,\ n=1,2)
\end{aligned} \tag{58}$$

Then, the integral equations (53) and (55) can be normalized as

$$\frac{1}{\pi}\int_{-1}^{1}\left\{\frac{\phi_2(r)}{r-s} + \sum_{j=1}^{2}[q_{1j}^{(1)}(s,r) + q_{2j}^{(1)}(s,r)]\phi_j(r)\right\}\mathrm{d}r = \frac{(1+\kappa)}{2m(s,0)}f_1(s) \tag{59}$$

$$\frac{1}{\pi}\int_{-1}^{1}\left\{\frac{\phi_1(r)}{r-s} + \sum_{j=1}^{2}[q_{1j}^{(2)}(s,r) + q_{2j}^{(2)}(s,r)]\phi_j(r)\right\}\mathrm{d}r = \frac{(1+\kappa)}{2m(s,0)}f_2(s) \tag{60}$$

The fundamental solution of these equations is of the form given by Golberg (1990) and Peters (1963)

$$w(r) = \frac{1}{\sqrt{1-r^2}} \tag{61}$$

and thus the unknowns $\phi_1(r)$ and $\phi_2(r)$ may be expressed in terms of Chebyshev polynomials of the first kind as follows:

$$\begin{aligned}
&\phi_1(r) = \frac{1}{\sqrt{1-r^2}}\sum_{n=0}^{N} c_n^{(1)}T_n(r) \quad -1 < r < 1 \\
&\phi_2(r) = \frac{1}{\sqrt{1-r^2}}\sum_{n=0}^{N} c_n^{(2)}T_n(r) \quad -1 < r < 1
\end{aligned} \tag{62}$$

where $c_n^{(1)}$ and $c_n^{(2)}$ $(n = 0,1,2\ldots)$ are unknown constants. Using the single valuedness condition (57) and considering the orthogonality conditions of $T_n(r)$, it can be shown that

$$\begin{aligned}
&c_0^{(1)} = 0 \\
&c_0^{(2)} = 0
\end{aligned} \tag{63}$$

Substituting (62) into (59) and (60), we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}c_n^{(1)}U_{n-1}(s)+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-1}^{1}\sum_{j=1}^{2}\left[q_{1j}^{(1)}(s,r)+q_{2j}^{(1)}(s,r)\right]c_n^{(j)}\frac{T_n(r)}{\sqrt{1-r^2}}\,\mathrm{d}r\\&\quad=\frac{(1+\kappa)}{2m(s,0)}f_1(s)\quad -1<s<1\\&\sum_{n=1}^{\infty}c_n^{(2)}U_{n-1}(s)+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-1}^{1}\sum_{j=1}^{2}\left[q_{1j}^{(2)}(s,r)+q_{2j}^{(2)}(s,r)\right]c_n^{(j)}\frac{T_n(r)}{\sqrt{1-r^2}}\,\mathrm{d}r\\&\quad=\frac{(1+\kappa)}{2m(s,0)}f_2(s)\quad -1<s<1\end{aligned}\tag{64}$$

Eq. (64) can be solved by truncating the series and choosing the collocation points s_n as

$$T_N(s_n)=0\quad s_n=\cos\left((2n-1)\frac{\pi}{2N}\right)\quad n=1,\ldots,N\tag{65}$$

After determining $c_n^{(1)}$ and $c_n^{(2)}$, the stress intensity factors at the crack tips may be expressed as (Konda and Erdogan, 1994)

$$\begin{aligned}k_1(a)&=\sqrt{\frac{b-a}{2}}\frac{2\mu(a,0)}{1+\kappa}\sum_{n=1}^{\infty}(-1)^n c_n^{(1)}\\k_2(a)&=\sqrt{\frac{b-a}{2}}\frac{2\mu(a,0)}{1+\kappa}\sum_{n=1}^{\infty}(-1)^n c_n^{(2)}\\k_1(b)&=-\sqrt{\frac{b-a}{2}}\frac{2\mu(b,0)}{1+\kappa}\sum_{n=1}^{\infty}c_n^{(1)}\\k_2(b)&=-\sqrt{\frac{b-a}{2}}\frac{2\mu(b,0)}{1+\kappa}\sum_{n=1}^{\infty}c_n^{(2)}\end{aligned}\tag{66}$$

and the crack surface openings as

$$\begin{aligned}u(x_1,+0)-u(x_1,-0)&=-\sqrt{(a'^2-x'^2)}\sum_{n=1}^{\infty}\frac{1}{n}c_n^{(1)}U_{n-1}(x'/a')\\v(x_1,+0)-v(x_1,-0)&=-\sqrt{(a'^2-x'^2)}\sum_{n=1}^{\infty}\frac{1}{n}c_n^{(2)}U_{n-1}(x'/a')\end{aligned}\tag{67}$$

where

$$a'=\frac{b-a}{2}\tag{68}$$

is the half-crack length and

$$x'=x-\frac{b+a}{2}\tag{69}$$

5. Results and discussion

Different crack lengths and orientations are used to calculate the stress intensity factors at the crack tips. In all cases, the loading is uniform strain at infinity, that is:

$$\varepsilon_{yy}(x, \pm\infty) = \varepsilon_0 \tag{70}$$

The crack surface tractions for this loading can be written as

$$\begin{aligned} p_1(x_1, 0) &= -\frac{8\mu_1 e^{\delta x \cos\theta}}{1+\kappa}\varepsilon_0 \cos^2\theta \\ p_2(x_1, 0) &= -\frac{8\mu_1 e^{\delta x \cos\theta}}{1+\kappa}\varepsilon_0 \cos\theta \sin\theta \end{aligned} \tag{71}$$

For various crack lengths, the calculations are carried out with θ varying from 0° to near 90°. All the stress intensity factors are normalized by

$$K_0 = \sigma_0\sqrt{a'} \tag{72}$$

where σ_0 is the normalizing stress and is defined as

$$\sigma_0 = \frac{8\mu_1}{1+\kappa}\varepsilon_0 \tag{73}$$

Fig. 2 shows the stress intensity factors for a crack with $a'/h = 0.05$ and $\delta = 0.03$. The solid lines indicate the stress intensity factors for an inclined crack in an FGM strip bonded to a homogeneous half-plane, while the dashed lines show the stress intensity factors for a crack in an FGM strip. As can be seen in Fig. 2, in this case, the difference between the two sets of results is negligible. This is largely due to the fact that the crack length is very small compared to the thickness of the strip. Thus, the perturbation brought upon by the homogeneous half-plane is not significant. However, as will be shown later, this effect will become more pronounced as the crack length increases.

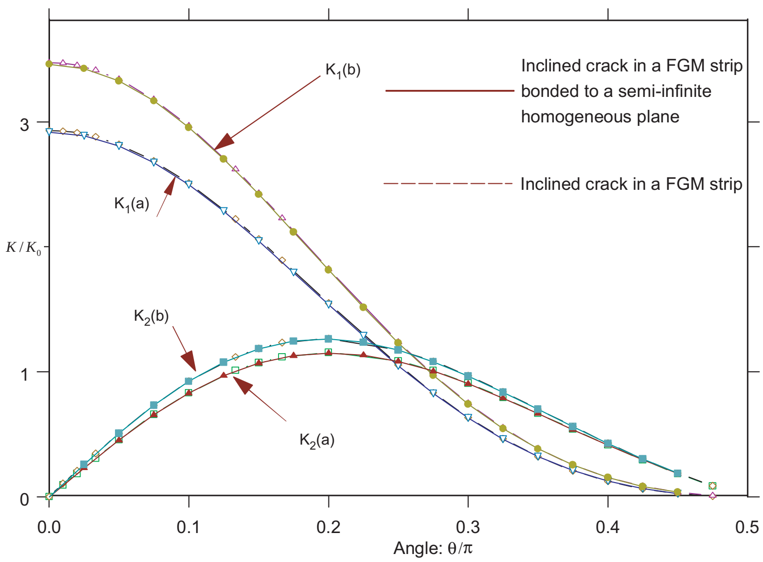

Fig. 2. Variation of the normalized stress intensity factors K/K_0 with θ/π for an embedded inclined crack in an FGM strip and an FGM strip bonded to a homogeneous half-plane under uniform strain, $a'/h = 0.05$.

Figs. 3–6 shows the stress intensity factors for cracks with varying crack lengths, for $a'/h = 0.10, 0.15, 0.20$ and 0.25, respectively. It can be observed that as the crack length increases, the difference between the stress intensity factors for an FGM strip and an FGM strip bonded to a homogeneous half-plane becomes more perceptible. But the trend of the variation of the intensity factors with respect to the crack angle remains the same.

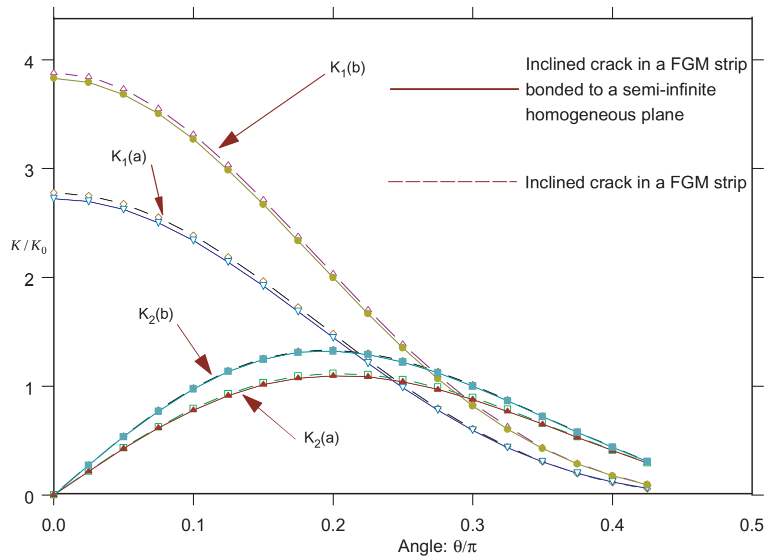

Fig. 3. Variation of the normalized stress intensity factors K/K_0 with θ/π for an embedded inclined crack in an FGM strip and an FGM strip bonded to a homogeneous half-plane under uniform strain, $a'/h = 0.10$.

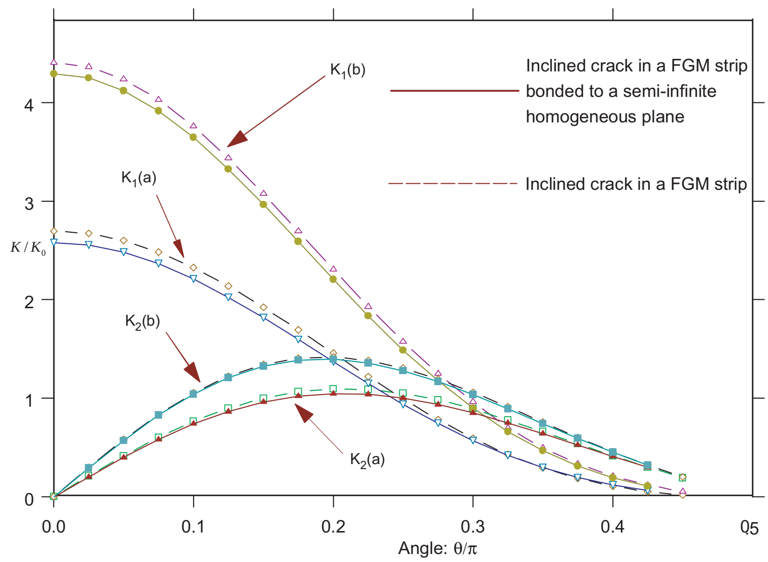

Fig. 4. Variation of the normalized stress intensity factors K/K_0 with θ/π for an embedded inclined crack in an FGM strip and an FGM strip bonded to a homogeneous half-plane under uniform strain, $a'/h = 0.15$.

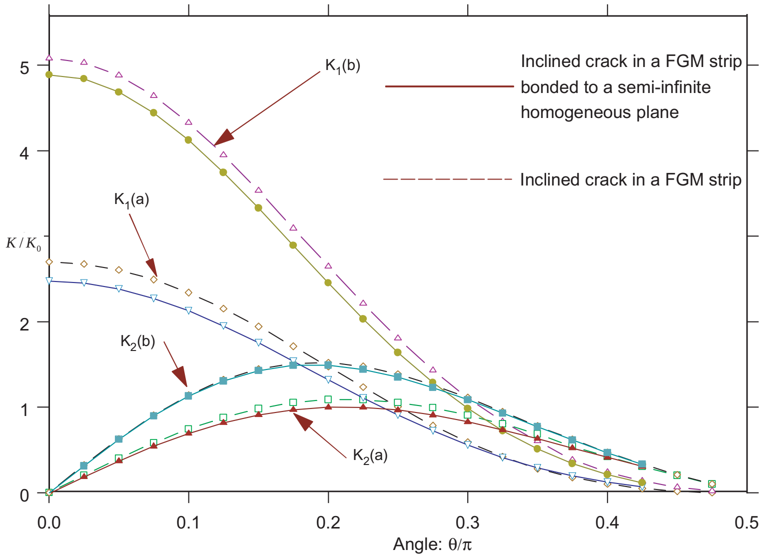

Fig. 5. Variation of the normalized stress intensity factors K/K_0 with θ/π for an embedded inclined crack in an FGM strip and an FGM strip bonded to a homogeneous half-plane under uniform strain, $a'/h = 0.20$.

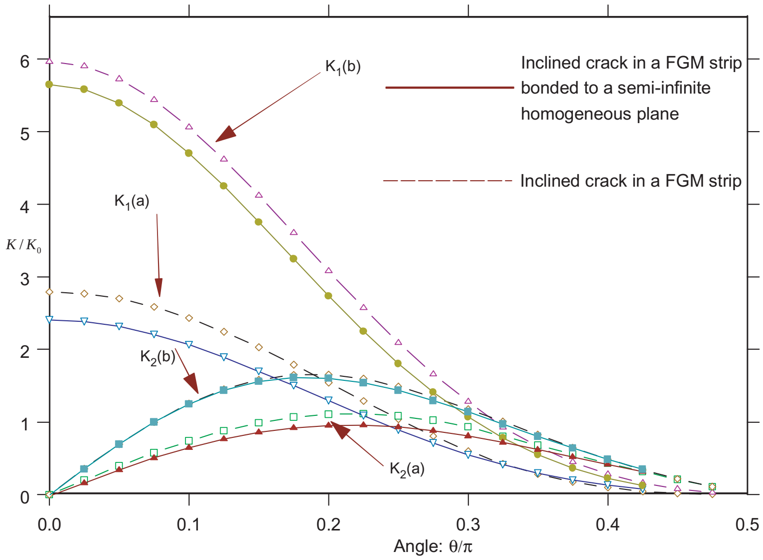

Fig. 6. Variation of the normalized stress intensity factors K/K_0 with θ/π for an embedded inclined crack in an FGM strip and an FGM strip bonded to a homogeneous half-plane under uniform strain, $a'/h = 0.25$.

From the results shown above, we can reach several important conclusions:

1. The square-root nature of the stress singularity is well maintained at the crack tips of cracks in FGM layer.
2. The stress intensity factors for mode *I* crack ($K_1(a)$ and$K_1(b)$) decrease when increasing θ, while the stress intensity factors for mode *II* crack, first increase and then decrease as the crack angle increases. The

stress intensity factors $(K_1)_s$ are always greater than $(K_2)_s$ in the beginning, when the problem is mostly under mode I deformation. After θ increases to a given point, $(K_1)_s$ become smaller than $(K_2)_s$, because mode *II* loading starts to dominate. This trend is not affected by the length of the crack.

3. Due to the existence of the nonhomogeneous nature of the material, for the parameter chosen the stress intensity factors increase significantly when the crack becomes longer.
4. In most cases, mode *I* fracture introduces larger stress concentration at crack tips. When mode II fracture dominates, the magnitude of the stress concentration is generally lower than that when mode I fracture dominates.
5. The homogeneous substrate affects the loading pattern of the crack, and consequently the stress intensity factors at crack tips. But its effect generally is negligible when the crack is small and away from interface, and it does not change the nature of the crack.

The crack surface openings are shown in Figs. 7 and 8. Fig. 7 depicts the crack surface opening in y_1 direction for the crack length $a'/h = 0.20$. Fig. 8 shows the corresponding opening in x_1 direction. In Fig. 8, it should be noted that there is no crack displacement in x_1 direction when $\theta = 0°$, thus, the orientations of the crack are chosen as 4.5°, 45° and 67.5°, respectively. In Fig. 7, the orientations of the crack are chosen as 0°, 45° and 67.5°. As expected, nonhomogeneity of the material increases the crack opening on the softer side of the material, while it reduces it on the stiffer side.

It is worth pointing out that although direct comparable experimental results are generally not available, some researchers have tried to solve the fracture problems of FGMs numerically. FEM is the typical method employed. Most of these studies are limited to cracks in a single FGM layer. Dolbow and Gosz (2002) computed mixed-mode stress intensity factors at the tips of arbitrarily oriented cracks in FGM, and the results were compared with the analytical solutions presented in the paper by Konda and Erdogan (1994). Good agreement was reported. Kim and Paulino (2002) gave a rather general finite element modeling of fracture in FGMs, with many interesting numerical results reported in the paper which were compared with the analytical solutions presented by Erdogan and Wu (1997). Those results are not directly

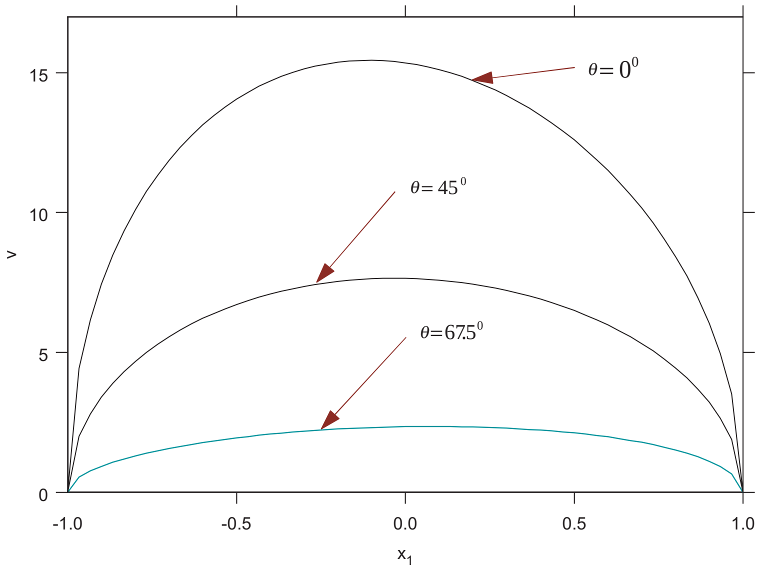

Fig. 7. Crack surface openings in the y_1 direction for $a'/h = 0.20$, $\theta = 0°$, 45° and 67.5°.

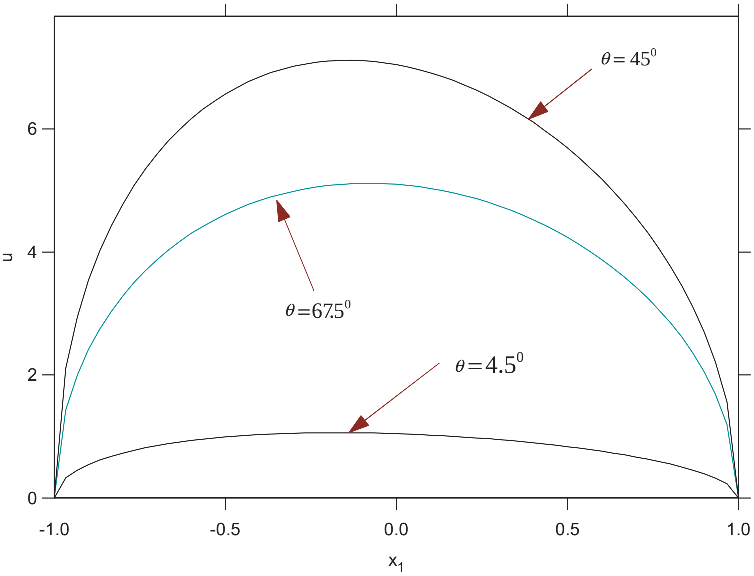

Fig. 8. Crack surface openings in the x_1 direction for $a'/h = 0.20$, $\theta = 0°$, 45° and 67.5°.

comparable to the analytical solutions in this paper, but were used as benchmark tests for the preliminary study of this paper by Long and Delale being published in the International Journal of Fracture.

It should be noted that the numerical calculations are extremely time consuming. Obtaining one data point required nearly one week of computational time on a PC. Accordingly more extensive results are not presented. Given enough time one may compute the stress intensity factors for various values of δ, and under bending and shear loadings.

Appendix A

Expressions of functions defined in the text

$$\begin{aligned}
R_1(\alpha) &= \left\{(m_4 - m_1)[(1+\kappa)n_1n_4 + (3-\kappa)\alpha^2] + \mathrm{i}\alpha(n_4 - n_1)[1+\kappa-(3-\kappa)m_1m_4]\right\}/R_0 \\
R_2(\alpha) &= \left\{(m_4 - m_2)[(1+\kappa)n_2n_4 + (3-\kappa)\alpha^2] + \mathrm{i}\alpha(n_4 - n_2)[1+\kappa-(3-\kappa)m_2m_4]\right\}/R_0 \\
R_3(\alpha) &= -\left\{(m_3 - m_1)[(1+\kappa)n_1n_3 + (3-\kappa)\alpha^2] + \mathrm{i}\alpha(n_3 - n_1)[1+\kappa-(3-\kappa)m_1m_3]\right\}/R_0 \\
R_4(\alpha) &= -\left\{(m_3 - m_1)[(1+\kappa)n_1n_3 + (3-\kappa)\alpha^2] + \mathrm{i}\alpha(n_3 - n_1)[1+\kappa-(3-\kappa)m_1m_3]\right\}/R_0 \\
R_0(\alpha) &= (m_4 - m_3)[(1+\kappa)n_3n_4 + (3-\kappa)\alpha^2] + \mathrm{i}\alpha(n_4 - n_3)[1+\kappa-(3-\kappa)m_3m_4]
\end{aligned} \tag{74}$$

$$\begin{aligned}
f_{1j} &= n_3m_4m_j(n_4 - n_j) + n_4m_3m_j(n_j - n_3) + n_jm_3m_4(n_3 - n_4) \\
f_{2j} &= m_4m_j(n_4 - n_j) + m_3m_j(n_j - n_3) + m_3m_4(n_3 - n_4) \\
f_{3j} &= n_3m_3(n_4 - n_j) + n_4m_4(n_j - n_3) + n_jm_j(n_3 - n_4) \\
f_{4j} &= m_3(n_4 - n_j) + m_4(n_j - n_3) + m_j(n_3 - n_4) \quad j = 1, 2
\end{aligned} \tag{75}$$

$$\begin{aligned} f_{1j} &= -[n_1 m_2 m_j (n_2 - n_j) + n_2 m_1 m_j (n_j - n_1) + n_j m_1 m_2 (n_1 - n_2)] \\ f_{2j} &= -[m_2 m_j (n_2 - n_j) + m_1 m_j (n_j - n_1) + m_1 m_2 (n_1 - n_2)] \\ f_{3j} &= -[n_1 m_1 (n_2 - n_j) + n_2 m_2 (n_j - n_1) + n_j m_j (n_1 - n_2)] \\ f_{4j} &= -[m_1 (n_2 - n_j) + m_2 (n_j - n_1) + m_j (n_1 - n_2)] \quad j = 3, 4 \end{aligned} \tag{76}$$

$$\begin{aligned} K_{11}^{(1)}(y', \alpha) &= -\frac{\mathrm{i}\alpha m_1(1+\kappa) - n_1(3-\kappa)}{\alpha\omega_0(\kappa-1)}[\alpha(3-\kappa)f_{22} + i(1+\kappa)f_{32}]\mathrm{e}^{n_1 y'} \\ &\quad + \frac{\mathrm{i}\alpha m_2(1+\kappa) - n_2(3-\kappa)}{\alpha\omega_0(\kappa-1)}[\alpha(3-\kappa)f_{21} + \mathrm{i}(1+\kappa)f_{31}]\mathrm{e}^{n_2 y'} \\ K_{12}^{(1)}(y', \alpha) &= -\frac{\mathrm{i}\alpha m_1(1+\kappa) - n_1(3-\kappa)}{\alpha\omega_0(\kappa-1)}(\mathrm{i}f_{12} + \alpha f_{42})(1+\kappa)\mathrm{e}^{n_1 y'} \\ &\quad + \frac{\mathrm{i}\alpha m_2(1+\kappa) - n_2(3-\kappa)}{\alpha\omega_0(\kappa-1)}(\mathrm{i}f_{11} + \alpha f_{41})(1+\kappa)\mathrm{e}^{n_2 y'} \end{aligned} \tag{77}$$

$$\begin{aligned} K_{21}^{(1)}(y', \alpha) &= \frac{\mathrm{i}\alpha m_1(\kappa-3) + n_1(1+\kappa)}{\alpha\omega_0(\kappa-1)}[\alpha(3-\kappa)f_{22} + \mathrm{i}(1+\kappa)f_{32}]\mathrm{e}^{n_1 y'} \\ &\quad - \frac{\mathrm{i}\alpha m_2(\kappa-3) + n_2(1+\kappa)}{\alpha\omega_0(\kappa-1)}[\alpha(3-\kappa)f_{21} + \mathrm{i}(1+\kappa)f_{31}]\mathrm{e}^{n_2 y'} \\ K_{22}^{(1)}(y', \alpha) &= \frac{\mathrm{i}\alpha m_1(\kappa-3) + n_1(1+\kappa)}{\alpha\omega_0(\kappa-1)}(\mathrm{i}f_{12} + \alpha f_{42})(1+\kappa)\mathrm{e}^{n_1 y'} \\ &\quad - \frac{\mathrm{i}\alpha m_2(\kappa-3) + n_2(1+\kappa)}{\alpha\omega_0(\kappa-1)}(\mathrm{i}f_{11} + \alpha f_{41})(1+\kappa)\mathrm{e}^{n_2 y'} \end{aligned} \tag{78}$$

$$\begin{aligned} K_{31}^{(1)}(y', \alpha) &= \frac{(n_1 m_1 - \mathrm{i}\alpha)}{\alpha\omega_0}[\alpha(3-\kappa)f_{22} + \mathrm{i}(1+\kappa)f_{32}]\mathrm{e}^{n_1 y'} \\ &\quad - \frac{(n_2 m_2 - \mathrm{i}\alpha)}{\alpha\omega_0}[\alpha(3-\kappa)f_{21} + \mathrm{i}(1+\kappa)f_{31}]\mathrm{e}^{n_2 y'} \\ K_{32}^{(1)}(y', \alpha) &= \frac{(n_1 m_1 - \mathrm{i}\alpha)}{\alpha\omega_0}(if_{12} + \alpha f_{42})(1+\kappa)\mathrm{e}^{n_1 y'} \\ &\quad - \frac{(n_2 m_2 - \mathrm{i}\alpha)}{\alpha\omega_0}(\mathrm{i}f_{11} + \alpha f_{41})(1+\kappa)\mathrm{e}^{n_2 y'} \end{aligned} \tag{79}$$

$$\begin{aligned} K_{11}^{(2)}(y', \alpha) &= -\frac{\mathrm{i}\alpha m_3(1+\kappa) - n_3(3-\kappa)}{\alpha\omega_0(\kappa-1)}[\alpha(3-\kappa)f_{24} + \mathrm{i}(1+\kappa)f_{34}]\mathrm{e}^{n_3 y'} \\ &\quad + \frac{\mathrm{i}\alpha m_4(1+\kappa) - n_4(3-\kappa)}{\alpha\omega_0(\kappa-1)}[\alpha(3-\kappa)f_{23} + \mathrm{i}(1+\kappa)f_{33}]\mathrm{e}^{n_4 y'} \\ K_{12}^{(2)}(y', \alpha) &= -\frac{\mathrm{i}\alpha m_3(1+\kappa) - n_3(3-\kappa)}{\alpha\omega_0(\kappa-1)}(\mathrm{i}f_{14} + \alpha f_{44})(1+\kappa)\mathrm{e}^{n_3 y'} \\ &\quad + \frac{\mathrm{i}\alpha m_4(1+\kappa) - n_4(3-\kappa)}{\alpha\omega_0(\kappa-1)}(\mathrm{i}f_{13} + \alpha f_{43})(1+\kappa)\mathrm{e}^{n_4 y'} \end{aligned} \tag{80}$$

$$
\begin{aligned}
K_{21}^{(2)}(y',\alpha) &= \frac{\mathrm{i}\alpha m_3(\kappa-3)+n_3(1+\kappa)}{\alpha\omega_0(\kappa-1)}[\alpha(3-\kappa)f_{24}+\mathrm{i}(1+\kappa)f_{34}]\mathrm{e}^{n_3 y'} \\
&\quad - \frac{\mathrm{i}\alpha m_4(\kappa-3)+n_4(1+\kappa)}{\alpha\omega_0(\kappa-1)}[\alpha(3-\kappa)f_{23}+\mathrm{i}(1+\kappa)f_{33}]\mathrm{e}^{n_4 y'} \\
K_{22}^{(2)}(y',\alpha) &= \frac{\mathrm{i}\alpha m_3(\kappa-3)+n_3(1+\kappa)}{\alpha\omega_0(\kappa-1)}(\mathrm{i}f_{14}+\alpha f_{44})(1+\kappa)\mathrm{e}^{n_3 y'} \\
&\quad - \frac{\mathrm{i}\alpha m_4(\kappa-3)+n_4(1+\kappa)}{\alpha\omega_0(\kappa-1)}(\mathrm{i}f_{13}+\alpha f_{43})(1+\kappa)\mathrm{e}^{n_4 y'}
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
K_{31}^{(2)}(y',\alpha) &= \frac{(n_3 m_3-\mathrm{i}\alpha)}{\alpha\omega_0}[\alpha(3-\kappa)f_{24}+\mathrm{i}(1+\kappa)f_{34}]\mathrm{e}^{n_3 y'} \\
&\quad - \frac{(n_4 m_4-\mathrm{i}\alpha)}{\alpha\omega_0}[\alpha(3-\kappa)f_{23}+\mathrm{i}(1+\kappa)f_{33}]\mathrm{e}^{n_4 y'} \\
K_{32}^{(2)}(y',\alpha) &= \frac{(n_3 m_3-\mathrm{i}\alpha)}{\alpha\omega_0}(\mathrm{i}f_{14}+\alpha f_{44})(1+\kappa)\mathrm{e}^{n_3 y'} \\
&\quad - \frac{(n_4 m_4-\mathrm{i}\alpha)}{\alpha\omega_0}(\mathrm{i}f_{13}+\alpha f_{43})(1+\kappa)\mathrm{e}^{n_4 y'}
\end{aligned} \tag{82}
$$

$$
\begin{aligned}
&Q_{31}(\alpha,t) \\
&= -\int_{-\infty}^{\infty} \frac{-\mathrm{i}\rho m_1[(1+\kappa)\cos^2\theta+(3-\kappa)\sin^2\theta]+n_1[(3-\kappa)\cos^2\theta+(1+\kappa)\sin^2\theta]-2(n_1 m_1-\mathrm{i}\rho)(\kappa-1)\cos\theta\sin\theta}{\rho\omega_0(\kappa-1)(n_1\cos\theta-\mathrm{i}\rho\sin\theta+\mathrm{i}\alpha)} \\
&\quad \times[\rho(3-\kappa)f_{22}+\mathrm{i}(1+\kappa)f_{32}]\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&\quad + \int_{-\infty}^{\infty} \frac{-\mathrm{i}\rho m_2[(1+\kappa)\cos^2\theta+(3-\kappa)\sin^2\theta]+n_2[(3-\kappa)\cos^2\theta+(1+\kappa)\sin^2\theta]-2(n_2 m_2-\mathrm{i}\rho)(\kappa-1)\cos\theta\sin\theta}{\rho\omega_0(\kappa-1)(n_2\cos\theta-\mathrm{i}\rho\sin\theta+\mathrm{i}\alpha)} \\
&\quad \times[\rho(3-\kappa)f_{21}+\mathrm{i}(1+\kappa)f_{31}]\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&\quad + \int_{-\infty}^{\infty} \frac{-\mathrm{i}\rho m_3[(1+\kappa)\cos^2\theta+(3-\kappa)\sin^2\theta]+n_3[(3-\kappa)\cos^2\theta+(1+\kappa)\sin^2\theta]-2(n_3 m_3-\mathrm{i}\rho)(\kappa-1)\cos\theta\sin\theta}{\rho\omega_0(\kappa-1)(n_3\cos\theta-\mathrm{i}\rho\sin\theta+\mathrm{i}\alpha)} \\
&\quad \times[\rho(3-\kappa)f_{24}+\mathrm{i}(1+\kappa)f_{34}]\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&\quad - \int_{-\infty}^{\infty} \frac{-\mathrm{i}\rho m_4[(1+\kappa)\cos^2\theta+(3-\kappa)\sin^2\theta]+n_4[(3-\kappa)\cos^2\theta+(1+\kappa)\sin^2\theta]-2(n_4 m_4-\mathrm{i}\rho)(\kappa-1)\cos\theta\sin\theta}{\rho\omega_0(\kappa-1)(n_4\cos\theta-\mathrm{i}\rho\sin\theta+\mathrm{i}\alpha)} \\
&\quad \times[\rho(3-\kappa)f_{23}+\mathrm{i}(1+\kappa)f_{33}]\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho
\end{aligned} \tag{83}
$$

$$
\begin{aligned}
&Q_{32}(\alpha,t) \\
&= -\int_{-\infty}^{\infty} \frac{-\mathrm{i}\rho m_1[(1+\kappa)\cos^2\theta+(3-\kappa)\sin^2\theta]+n_1[(3-\kappa)\cos^2\theta+(1+\kappa)\sin^2\theta]-2(n_1 m_1-\mathrm{i}\rho)(\kappa-1)\cos\theta\sin\theta}{\rho\omega_0(\kappa-1)(n_1\cos\theta-\mathrm{i}\rho\sin\theta+\mathrm{i}\alpha)} \\
&\quad \times(\mathrm{i}f_{12}+\rho f_{42})(1+\kappa)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&\quad + \int_{-\infty}^{\infty} \frac{-\mathrm{i}\rho m_2[(1+\kappa)\cos^2\theta+(3-\kappa)\sin^2\theta]+n_2[(3-\kappa)\cos^2\theta+(1+\kappa)\sin^2\theta]-2(n_2 m_2-\mathrm{i}\rho)(\kappa-1)\cos\theta\sin\theta}{\rho\omega_0(\kappa-1)(n_2\cos\theta-\mathrm{i}\rho\sin\theta+\mathrm{i}\alpha)} \\
&\quad \times(\mathrm{i}f_{11}+\rho f_{41})(1+\kappa)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&\quad + \int_{-\infty}^{\infty} \frac{-\mathrm{i}\rho m_3[(1+\kappa)\cos^2\theta+(3-\kappa)\sin^2\theta]+n_3[(3-\kappa)\cos^2\theta+(1+\kappa)\sin^2\theta]-2(n_3 m_3-\mathrm{i}\rho)(\kappa-1)\cos\theta\sin\theta}{\rho\omega_0(\kappa-1)(n_3\cos\theta-\mathrm{i}\rho\sin\theta+\mathrm{i}\alpha)} \\
&\quad \times(\mathrm{i}f_{14}+\rho f_{44})(1+\kappa)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&\quad - \int_{-\infty}^{\infty} \frac{-\mathrm{i}\rho m_4[(1+\kappa)\cos^2\theta+(3-\kappa)\sin^2\theta]+n_4[(3-\kappa)\cos^2\theta+(1+\kappa)\sin^2\theta]-2(n_4 m_4-\mathrm{i}\rho)(\kappa-1)\cos\theta\sin\theta}{\rho\omega_0(\kappa-1)(n_4\cos\theta-\mathrm{i}\rho\sin\theta+\mathrm{i}\alpha)} \\
&\quad \times(\mathrm{i}f_{13}+\rho f_{43})(1+\kappa)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
Q_{41} = &-\int_{-\infty}^{\infty} \frac{-\sin 2\theta(\mathrm{i}\rho m_1 + n_1) + (n_1 m_1 - \mathrm{i}\rho)\cos 2\theta}{\rho\omega_0(n_1\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)}[\rho(3-\kappa)f_{22} + \mathrm{i}(1+\kappa)f_{32}]\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&+\int_{-\infty}^{\infty} \frac{-\sin 2\theta(\mathrm{i}\rho m_2 + n_2) + (n_2 m_2 - \mathrm{i}\rho)\cos 2\theta}{\rho\omega_0(n_2\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)}[\rho(3-\kappa)f_{21} + \mathrm{i}(1+\kappa)f_{31}]\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&+\int_{-\infty}^{\infty} \frac{-\sin 2\theta(\mathrm{i}\rho m_3 + n_3) + (n_3 m_3 - \mathrm{i}\rho)\cos 2\theta}{\rho\omega_0(n_3\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)}[\rho(3-\kappa)f_{24} + \mathrm{i}(1+\kappa)f_{34}]\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&-\int_{-\infty}^{\infty} \frac{-\sin 2\theta(\mathrm{i}\rho m_4 + n_4) + (n_4 m_4 - \mathrm{i}\rho)\cos 2\theta}{\rho\omega_0(n_4\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)}[\rho(3-\kappa)f_{23} + \mathrm{i}(1+\kappa)f_{33}]\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho
\end{aligned}
\tag{85}
$$

$$
\begin{aligned}
Q_{42} = &-\int_{-\infty}^{\infty} \frac{-\sin 2\theta(\mathrm{i}\rho m_1 + n_1) + (n_1 m_1 - \mathrm{i}\rho)\cos 2\theta}{\rho\omega_0(n_1\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)}(\mathrm{i}f_{12} + \rho f_{42})(1+\kappa)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&+\int_{-\infty}^{\infty} \frac{-\sin 2\theta(\mathrm{i}\rho m_2 + n_2) + (n_2 m_2 - \mathrm{i}\rho)\cos 2\theta}{\rho\omega_0(n_2\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)}(\mathrm{i}f_{11} + \rho f_{41})(1+\kappa)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&\times\int_{-\infty}^{\infty} \frac{-\sin 2\theta(\mathrm{i}\rho m_3 + n_3) + (n_3 m_3 - \mathrm{i}\rho)\cos 2\theta}{\rho\omega_0(n_3\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)}(\mathrm{i}f_{14} + \rho f_{44})(1+\kappa)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho \\
&-\int_{-\infty}^{\infty} \frac{-\sin 2\theta(\mathrm{i}\rho m_4 + n_4) + (n_4 m_4 - \mathrm{i}\rho)\cos 2\theta}{\rho\omega_0(n_4\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)}(\mathrm{i}f_{13} + \rho f_{43})(1+\kappa)\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
Q_{51}(\alpha,t) = &\frac{1}{2\pi(1+\kappa)}\int_{-\infty}^{\infty}\left\{-\frac{(\sin\theta - m_1\cos\theta)[\rho(3-\kappa)f_{22} + \mathrm{i}(1+\kappa)f_{32}]}{(n_1\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0}\right. \\
&+\frac{(\sin\theta - m_2\cos\theta)[\rho(3-\kappa)f_{21} + \mathrm{i}(1+\kappa)f_{31}]}{(n_2\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0} + \frac{(\sin\theta - m_3\cos\theta)[\rho(3-\kappa)f_{24} + \mathrm{i}(1+\kappa)f_{34}]}{(n_3\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0} \\
&\left.-\frac{(\sin\theta - m_4\cos\theta)[\rho(3-\kappa)f_{23} + \mathrm{i}(1+\kappa)f_{33}]}{(n_4\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0}\right\}\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
Q_{52}(\alpha,t) = &\frac{1}{2\pi(1+\kappa)}\int_{-\infty}^{\infty}\left\{-\frac{(\sin\theta - m_1\cos\theta)(if_{12} + \rho f_{42})(1+\kappa)}{(n_1\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0}\right. \\
&+\frac{(\sin\theta - m_2\cos\theta)(\mathrm{i}f_{11} + \rho f_{41})(1+\kappa)}{(n_2\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0} + \frac{(\sin\theta - m_3\cos\theta)(\mathrm{i}f_{14} + \rho f_{44})(1+\kappa)}{(n_3\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0} \\
&\left.-\frac{(\sin\theta - m_4\cos\theta)(\mathrm{i}f_{13} + \rho f_{43})(1+\kappa)}{(n_4\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0}\right\}\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
Q_{61}(\alpha,t) = &\frac{1}{2\pi(1+\kappa)}\int_{-\infty}^{\infty}\left\{-\frac{(m_1\sin\theta + \cos\theta)[\rho(3-\kappa)f_{22} + \mathrm{i}(1+\kappa)f_{32}]}{(n_1\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0}\right. \\
&+\frac{(m_2\sin\theta + \cos\theta)[\rho(3-\kappa)f_{21} + \mathrm{i}(1+\kappa)f_{31}]}{(n_2\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0} + \frac{(m_3\sin\theta + \cos\theta)[\rho(3-\kappa)f_{24} + \mathrm{i}(1+\kappa)f_{34}]}{(n_3\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0} \\
&\left.-\frac{(m_4\sin\theta + \cos\theta)[\rho(3-\kappa)f_{23} + \mathrm{i}(1+\kappa)f_{33}]}{(n_4\cos\theta - \mathrm{i}\rho\sin\theta + \mathrm{i}\alpha)\rho\omega_0}\right\}\mathrm{e}^{\mathrm{i}\rho t}\,\mathrm{d}\rho
\end{aligned}
\tag{89}
$$

$$
\begin{aligned}
Q_{62}(\alpha, t) = {} & \frac{1}{2\pi(1+\kappa)} \int_{-\infty}^{\infty} \left\{ -\frac{(m_1 \sin\theta + \cos\theta)(\mathrm{i}f_{12} + \rho f_{42})(1+\kappa)}{(n_1 \cos\theta - \mathrm{i}\rho \sin\theta + \mathrm{i}\alpha)\rho\omega_0} \right. \\
& + \frac{(m_2 \sin\theta + \cos\theta)(\mathrm{i}f_{11} + \rho f_{41})(1+\kappa)}{(n_2 \cos\theta - \mathrm{i}\rho \sin\theta + \mathrm{i}\alpha)\rho\omega_0} + \frac{(m3_j \sin\theta + \cos\theta)(\mathrm{i}f_{14} + \rho f_{44})(1+\kappa)}{(n_3 \cos\theta - \mathrm{i}\rho \sin\theta + \mathrm{i}\alpha)\rho\omega_0} \\
& \left. - \frac{(m_4 \sin\theta + \cos\theta)(\mathrm{i}f_{13} + \rho f_{43})(1+\kappa)}{(n_4 \cos\theta - \mathrm{i}\rho \sin\theta + \mathrm{i}\alpha)\rho\omega_0} \right\} e^{\mathrm{i}\rho t}\, d\rho
\end{aligned} \tag{90}
$$

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